**Formulation of Documents**

This file introduces the formulation of each element. We will first introduce the relationship between strain energy, internal forces, and stiffness of general deformable bodies. While introducing the basic theory, we will discuss how to compute internal forces and stiffness matrix using central difference method. After that we will derive each element and show how to code these elements in MATLAB script.

**Strain Energy, Internal Forces, Stiffness Matrix, and Central Difference**

Any deformable body is like a spring – if we deform it by applying external forces, it will store a strain energy. When we release the applied force, the store energy will be released. If the deformation is “elastic”, the body will return to its original state, just like a spring. If the body does not recover to its original state, “non-elastic” deformation happens. Now, consider a spring with linear elastic behavior. Our high school physics knowledge tells us that the internal force of this spring is linearly proportional to the extension , following the Hook law:

where the spring stiffness is . Then, the strain energy of this spring can be written as:

The strain energy is the shaded area under the *F*-*u* curve, and it gives the energy stored within the spring when it is stretched to an extension . We can see from the above analysis that there is some relationship between the potential energy, internal forces, and stiffness - the internal force is the derivative of the strain energy:

and the stiffness is the derivative of internal force:

Nothing surprising right! Now, let’s consider another situation, instead of representing the spring using an extension . Consider that the spring is in a 2D planar space, and we want to represent it using the two nodal coordinates and of this spring. This is helpful because we can use the nodal coordinates to directly locate the spring. If this spring is part of an active structure, it could be everywhere. Assuming that the spring has an original length , the extension of the spring can be calculated using the nodal coordinates:

Here, we can write the nodal coordinates of this spring as a vector . The potential of this spring element now becomes:

Now, because the total potential is a function of a 4 by 1 vector, the internal force can be solved as the Jacobian of the potential and is also a 4 by 1 vector:

This result looks very similar – it is just like a 2D truss element we learned in fundamental mechanics classes. The magnitude of the internal force is , while the rest of the term rotate the internal forces into direction along the spring. The first two rows are forces acting on node 1 of this spring, while the second two rows are forces acting on node 2 of this spring.

The stiffness matrix of this spring can also be solved using the same procedure as before. In this case, the stiffness matrix of this spring is the Hessian matrix of the total potential and it has the form:

Formulation of Bar Elements